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Stimulated Emission of Relativistic Particles by External Sources in Spacetime

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Explicit solutions are derived for transition amplitudes associated with stimulated emission of relativistic particles by external sources in *spacetime*. More precisely, exact expressions are obtained for transition amplitudes for any process where there are initially, at a given time, an arbitrary number of particles localized in various regions of space, prior to the switching on of an intervening source, and then, finally, at a later time when the intervening source ceases to operate, a given number of particles are found to be localized in various regions of space. The analysis is given for massive particles of *arbitrary* integer and half-integer spins. The solutions are obtained by carrying out a unitarity expansion in *configuration* space, where particles travel between emitters and detectors in the presence of an intervening source. Considered as an application is the process: particle → arbitrary number of particles, where the latter particles emerge spatially with a cone.

1. INTRODUCTION

In spite of the many experiments [see, e.g., Franson and Potocki (1988), Grangier *et al.* (1986), and Grishaev *et al.* (1971), as well as the pioneering work of Taylor (1909) at the beginning of the century] giving a clear indication that not only massive particles but also single photons may be localized by detectors, almost no consistent and systematic studies seem to have been carried out to formulate the language in which actual *computations* of physical processes involving relativistic particles in quantum field theory in *spacetime* may be worked out. By the computations of physical processes, I mean the calculation of transition probabilities of collisions or decay processes where, say, the final products emerge *spatially* into various cones. Much of the earlier efforts [see Han *et al.* (1987) and Ali (1985) and the pioneering work of Newton and Wigner (1949)] on the localization problem

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of relativistic particles dealt with so-called wave functions (as done in nonrelativistic theory) and position operators(!) and are so remote from the actual physical problem of the propagation of the particles in spacetime (cf. Feynman, 1949) between emitters and detectors that they offer no hope of treating particle-particle interacting theories in spacetime. Clearly, such a spacetime description of interacting theories seems possible only within the language of quantum field theory (Manoukian, 1988). The present paper gives a systematic study for the treatment of physical processes of relativistic particles in quantum field theory in a spacetime analysis of stimulated emission by external sources. Unfortunately, the transition from momentum space (cf. Manoukian, 1986*a*) to spacetime is far from obvious and the solutions for the latter are obtained by satisfying the severe completeness relation of a unitarity expansion in *configuration* space for the propagation of relativistic particles from an emitter to an intervening source and finally to a detector. (The latter reference, however, is indispensable for the present study.) The situation involving stimulated emission is clear. Initially, at a given time y_1^0 we have a certain number of particles localized in various regions of space. At a later time, an external (intervening) source is switched on which may absorb some or all of these particles as they move, stimulating further emissions by the source. At a later time y_2^0, after the intervening source ceases to operate, we again have a certain number of particles in various regions of space. Apart from the physical problem of stimulated emission in its own right, the intervening sources mimic various possible particle-particle interactions, and earlier methods (Manoukian, 1986*b*, 1989) give the hope of developing a tool for computations of physical processes, such as transition probabilities, involving relativistic particles in quantum field theory in spacetime.

This paper deals with stimulated emission of massive relativistic particles of *arbitrary* spins in space time. Massless particles and particle-particle interactions (cf. Manoukian, 1986*b*, 1989) in spacetime will be the subject of a forthcoming report. Section 2 considers stimulated emission of spin-0 particles in spacetime and derives the exact corresponding amplitudes. Section 3 generalizes the analysis of Section 2 first to spin-1/2 particles and then to arbitrary spins $s > 0$. In the final section (Section 4), some examples are given and an explicit spacetime computation of the transition probability for the decay process particle → arbitrary numer of particles is worked out, where the latter particles emerge *spatially* within a cone. By invoking the classic Araki–Haag–Ruelle theorem (Araki, 1962; Ruelle, 1962; Dollard and Velo, 1966) of estimates of smooth solutions of the Klein–Gordon equation, the decaying particles are seen to have, asymptotically in time, the direction of their momenta within the same cone, as expected on physical grounds.

2. STIMULATED EMISSION IN SPACETIME: SPIN-0 PARTICLES

The starting point is Schwinger's vacuum-to-vacuum transition amplitude (Schwinger, 1970; Manoukian, 1984) for chargeless, spin-0 particles interacting with an external source $K(x)$. The latter is given by the expression

$$\langle 0_+|0_-\rangle^K = \exp\frac{i}{2}\int (dx)\,(dx')\,K(x)\Delta_+(x-x')K(x') \tag{1}$$

where

$$\Delta_+(x-x') = i\int \frac{d^3\mathbf{k}}{(2\pi)^3 2k^0}\,e^{ik(x-x')} \qquad \text{for} \quad x^0 > x'^0 \tag{2}$$

$k^0 = (\mathbf{k}^2 + m^2)^{1/2}$. We define the function of time-space $y = (y^0, \mathbf{y})$

$$a(y) = \int \frac{d^3\mathbf{k}}{(2\pi)^3(2k^0)^{1/2}}\,e^{iky}K(k) \tag{3}$$

Then the vacuum persistence probability may be written as

$$|\langle 0_+|0_-\rangle^K|^2 = \exp\left[-\int d^3\mathbf{y}\,|a(y)|^2\right] \leq 1 \tag{4}$$

For the subsequent analysis, it is convenient to introduce a discrete (Schwinger, 1970; Manoukian, 1984) space variable notation (a lattice) by setting in the process

$$a_y = (d^3\mathbf{y})^{1/2} a(y) \tag{5}$$

$y = (y^0, \mathbf{y})$. Let $\{\mathbf{y}_1, \mathbf{y}_2, \ldots\}$ denote the set of the lattice points; then we may rewrite (4) as

$$|\langle 0_+|0_-\rangle^K|^2 = \exp\left[-\sum_{\mathbf{y}} |a_y|^2\right] \tag{6}$$

To obtain the transition amplitudes for stimulated emission we proceed as follows. We write (c.f. Manoukian, 1986*a*) $K = K_1 + K_2 + K_3$, where the source K_2 is switched on after the source K_1 is switched off, and the source K_3 is switched on after the source K_2 is switched off. The source K_2 is called an intervening source. After straightforward manipulations, we may rewrite (1) as

$$\begin{aligned}\langle 0_+|0_-\rangle^K &= \langle 0_+|0_-\rangle^{K_3}\langle 0_+|0_-\rangle^{K_2}\langle 0_+|0_-\rangle^{K_1} \exp ia^{3*}ia^2 \\ &\quad \times \exp ia^{3*}\tilde{\delta} ia^1 \exp ia^{2*}ia^1\end{aligned} \tag{7}$$

where

$$a^{3*}a^{2}=\sum_{\mathbf{y}} a_y^{3*}a_y^2 \tag{8}$$

$$a^{2*}a^{1}=\sum_{\mathbf{y}} a_y^{2*}a_y^1 \tag{9}$$

and

$$a^{3*}\tilde{\delta}a^{1}=\sum_{\mathbf{y}}\sum_{\mathbf{y}'} a_y^{3*}\tilde{\delta}_{yy'}a^1_{y'} \tag{10}$$

$$\tilde{\delta}_{yy'}=(d^3\mathbf{y}\, d^3\mathbf{y}')^{1/2}\int d^3\mathbf{x}\left[\Delta(y-x)i\frac{\overleftrightarrow{\partial}}{\partial x^0}\Delta(x-y')\right] \tag{11}$$

$y=(y_2^0,\mathbf{y})$, $y'=(y_1^0,\mathbf{y}')$, $y_2^0>y_1^0$; y_1^0 is chosen to be any time after the source K_1 is switched off and before K_2 is switched on, and y_2^0 is chosen to be any time after K_2 is switched off and before K_3 is switched on;

$$\Delta(y-x)=\int\frac{d^3\mathbf{k}}{(2\pi)^3(2k^0)^{1/2}}\,e^{ik(y-x)} \tag{12}$$

$$\overleftrightarrow{\partial}/\partial x^0=\overrightarrow{\partial}/\partial x^0-\overleftarrow{\partial}/\partial x^0$$

Upon expanding the exponentials in (7), one can rewrite the latter in detail as

$$\begin{aligned}\langle 0_+|0_-\rangle^{K}&=\langle 0_+|0_-\rangle^{K_3}\sum{}^{*}\frac{(ia^{3*}_{y_1})^{N_1}}{(N_1!)^{1/2}}\frac{(ia^{3*}_{y_2})^{N_2}}{(N_2!)^{1/2}}\\&\quad\times\cdots[\cdot]^{K_2}\frac{(ia^{2}_{y_1})^{M_1}}{(M_1!)^{1/2}}\frac{(ia^{2}_{y_2})^{M_2}}{(M_2!)^{1/2}}\cdots\langle 0_+|0_-\rangle^{K_1}\end{aligned} \tag{13}$$

where

$$\begin{aligned}[\cdot]^{K_2}&=\langle 0_+|0_-\rangle^{K_2}(N_1!\,N_2!\ldots M_1!\,M_2!\ldots)^{1/2}\frac{(ia^{2}_{y_1})^{N_1-m_1}}{(N_1-m_1)!}\frac{(ia^{2}_{y_2})^{N_2-m_2}}{(N_2-m_2)!}\\&\quad\times\cdots\frac{(\tilde{\delta}_{y_1y_1'})^{m_{11}}}{m_{11}!}\frac{(\tilde{\delta}_{y_1y_2'})^{m_{12}}}{m_{12}!}\cdots\frac{(\tilde{d}_{y_2y_1'})^{m_{21}}}{m_{21}!}\frac{(\tilde{\delta}_{y_2y_2'})^{m_{22}}}{m_{22}!}\\&\quad\times\cdots\frac{(ia^{2*}_{y_1'})^{M^1=\sum_i m_{i1}}}{(M_i-\sum_i m_{i1})!}\frac{(ia^{2*}_{y_2'})^{M_2-\sum_i m_{i2}}}{(M_2-\sum_i m_{i2})!}\end{aligned} \tag{14}$$

and $\sum^*$ stands for a summation over all nonegative integers: N; M; N_1, $N_2,\ldots$; M_1, $M_2,\ldots$; such that $N_1+N_2+\cdots=N$, $M_1+M_2+\cdots=M$; as well as over all nonnegative integers: m; m_1;

$m_2, \ldots; m_{11}, m_{12}, \ldots, m_{21}, m_{22}, \ldots;$ satisfying the constraints

$$\begin{aligned} m_{11}+m_{12}+\cdots &= m_1, \qquad 0 \le m_{11}+m_{21}+\cdots \le M_1 \\ m_{21}+m_{22}+\cdots &= m_2, \qquad 0 \le m_{12}+m_{22}+\cdots \le M_2 \\ &\ \vdots \qquad\qquad\qquad \vdots \end{aligned} \tag{15}$$

$$\begin{aligned} 0 \le\ & m_1 \le N_1 \\ 0 \le\ & m_2 \le N_2 \\ & \vdots \end{aligned} \tag{16}$$

$$m_1+m_2+\cdots = m \tag{17}$$

Finally, $y_i = (y_2^0, \mathbf{y}_i)$, $y_i' = (y_1^0, \mathbf{y}_i)$, where the times y_2^0, y_1^0 have been introduced just below equation (11). A unitarity expansion may be also carried out for $\langle 0_+ | 0_- \rangle^K$ in configuration space as follows:

$$\begin{aligned} \langle 0_+ | 0_- \rangle^K = \textstyle\sum_* & \langle 0_+ | N; N_1, N_2, \ldots, y_2^0 \rangle^{K_3} \\ & \times \langle N; N_1, N_2, \ldots, y_2^0 | M; M_1, M_2, \ldots, y_1^0 \rangle^{K_2} \\ & \times \langle M, M_1, M_2, \ldots, y^0 | 0_- \rangle^{K_1} \end{aligned} \tag{18}$$

where $y_2^0 > y_1^0$, $\langle M, M_1, M_2, \ldots, y_1^0 | 0_- \rangle^{K_1}$ denotes the amplitude that M particles are emitted by the source K_1, M_1 of which are found at lattice site $\mathbf{y}_1$, M_2 of which are found at $\mathbf{y}_2$, and so on, *at a time* y_1^0 after the source K_1 ceases to operate. Here

$$\langle N; N_1, N_2, \ldots, y_2^0 | M; M_1, M_2, \ldots, y_1^0 \rangle^{K_2}$$

denotes the amplitude that M particles, M_1 of which were at lattice site $\mathbf{y}_1$, M_2 of which were at $\mathbf{y}_2$, and so on, initially at time y_1^0, move in the presence of the intervening source K_2, and at a later time y_2^0, after the latter source ceases to operate, we find N particles, N_1 of which are at lattice site $\mathbf{y}_1$, N_2 of which are at lattice site $\mathbf{y}_2$, and so on. The latter amplitude is the object of interest. It gives the amplitude for having finally in a process N particles after the intervening source is switched off when there are initially N particles before the intervening source is switched on. It is the amplitude of stimulated emission of particles. Finally, $\langle 0_+ | N; N_1, N_2, \ldots, y_2^0 \rangle^{K_3}$ is the amplitude that N particles are absorbed by K_3 when at a time y_2^0 before the source K_3 was switched on, N_1 of the particles were at lattice site $\mathbf{y}_1$, N_2 at lattice site $\mathbf{y}_2$, and so on. $\sum_*$ stands for a summation over all nonnegative integers N; $N_1, N_2, \ldots$; M; $M_1, M_2, \ldots$; such that $N_1+N_2+\cdots = N$, $M_1+M_2+\cdots = M$.

By setting first $K_2 = 0$, we have from (18) and (13)

$$\langle M; M_1, M_2, \ldots, y^0 | 0_- \rangle^{K_1} = \langle 0_+ | 0_- \rangle^{K_1} \frac{(ia_{y_1})^{M_1}}{(M_1!)^{1/2}} \frac{(ia_{y_2})^{M_2}}{(M_2!)^{1/2}} \cdots \tag{19}$$

$$\langle 0_+|N; N_1, N_2, \ldots, y^0\rangle^{K_3} = \langle 0_+|0_-\rangle^{K_3} \frac{(ia^*_{y_1})^{N_1}}{(N_1!)^{1/2}} \frac{(ia^*_{y_2})^{N_2}}{(N_2!)^{1/2}} \cdots \tag{20}$$

$y_i = (y^0, \mathbf{y}_i)$, where y^0 is arbitrary, falling in the time interval before the source K_3 is switched on and after the source K_1 is switched off. Quite generally, for $K_2 \neq 0$, we may extract the expression for the stimulated emission transition amplitude from (13) and (18) to be

$$\langle N; N_1, N_2, \ldots y_2^0 | M; M_1, M_2, \ldots, y_1^0\rangle^K = [\cdot]^K \tag{21}$$

for a given intervening source K, where $[\cdot]^K$ is defined in (14) upon setting $K_2 = K$ in the latter. Equation (21) is the main contribution of this paper. An explicit application wil be given in Section 4.

3. GENERALIZATION TO ARBITRARY SPINS

We consider first spin-1/2 particles. To this end, the vacuum-to-vacuum transition amplitude in the presence of external (anticommuting) sources $\eta(x)$, $\bar{\eta}(x)$ may be written in the form

$$\langle 0_+|0_-\rangle^\eta = \exp i \int (dx)\,(dx')\; \bar{\eta}(x) S_+(x-x')\eta(x') \tag{22}$$

where

$$S_+(x-x') = i \int \frac{d^3\mathbf{p}}{(2\pi)^3 2p^0} (-\gamma p + m) e^{ip(x-x')} \qquad \text{for} \quad x^0 > x'^0 \tag{23}$$

$p^0 = (\mathbf{p}^2 + m^2)^{1/2}$. As in Section 2, we write $\eta = \eta_1 + \eta_2 + \eta_3$, to obtain for (22)

$$\begin{aligned} \langle 0_+|0_-\rangle^\eta = \langle 0_+|0_-\rangle^{\eta_3} \langle 0_+|0_-\rangle^{\eta_2} \langle 0_+|0_-\rangle^{\eta_1} \\ \times \exp iU_3^* iU_2 \exp iU_3^* \tilde{\delta} iU_1 \\ \times \exp iU_2^* iU_1 \end{aligned} \tag{24}$$

where

$$U_3^* U_2 = \sum_{\sigma,r} \int d^3\mathbf{x}\; U_3(x, \sigma, r)^* U_2(x, \sigma, r) \tag{25}$$

$$U_2^* U_1 = \sum_{\sigma,r} \int d^3\mathbf{x}\; U_2(x, \sigma, r)^* U_1(x, \sigma, r) \tag{26}$$

$$U_3^* \tilde{\delta} U_1 = \sum_{\substack{\sigma,\sigma' \\ r,r'}} \int d^3\mathbf{y}_2 \int d^3\mathbf{y}_1\; U_3(y_2, \sigma, r)^* \tilde{\delta}^{rr'}_{\sigma\sigma'}(y_2 - y_1) U_1(y_1, \sigma', r') \tag{27}$$

$$\tilde{\delta}^{rr'}_{\sigma\sigma'}(y_2 - y_1) = \delta^{rr'} \delta_{\sigma\sigma'} \tilde{\delta}(y_2 - y_1) \tag{28}$$

and $\tilde{\delta}$ [refer to equation (11)]

$$\tilde{\delta}(y_2 - y_1) = \int d^3\mathbf{x}\, \Delta(y_2 - x) \frac{i\overleftrightarrow{\partial}}{\partial x^0} \Delta(x - y_1) \tag{29}$$

$$y_2^0 > x^0 > y_1^0$$

$$U^*(x, \sigma, +) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left(\frac{m}{p^0}\right)^{1/2} \bar{\eta}(p) u(p, \sigma) e^{-ipx} \tag{30}$$

$$U(x, \sigma, +) = \int \frac{d^3\mathbf{p}}{(2\pi)^2} \left(\frac{m}{p^0}\right)^{1/2} \bar{u}(p, \sigma) \eta(p) e^{ipx} \tag{31}$$

$$U^*(x, \sigma, -) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left(\frac{m}{p^0}\right)^{1/2} \bar{v}(p, \sigma) \eta(-p) e^{-ipx} \tag{32}$$

$$U(x, \sigma, -) = \int \frac{d^3\mathbf{p}}{(2\pi p)^3} \left(\frac{m}{p^0}\right)^{1/2} \bar{\eta}(-p) v(p, \sigma) e^{ipx} \tag{33}$$

where $r = \pm$ corresponds to particle, antiparticle, respectively. $\sigma = \pm 1/2$ is a mere labeling, corresponding to spin values assigned to the spinors $u(p, \sigma)$, $v(p, \sigma)$. Note that $(U(x, \sigma, r))^2 = 0$.

We introduce the notation $\alpha = (y_2^0, \mathbf{y}, \sigma, r)$, $\alpha' = (y_1^0, \mathbf{y}, \sigma, r)$, and a convenient discrete notation for the space variable (a lattice), and introduce in the process the notation

$$U_\alpha = (d^3\mathbf{y})^{1/2}\, U(y_2^0, \mathbf{y}, \sigma, r) \tag{34}$$

We consider only connected processes, where all the particles initially present are detected (absorbed) by the intervening source η_2, deleting all those processes where some or all the initial particles just move from the emitter to the detector without being detected by the intervening source. A similar analysis as in Section 2 then readily shows [see also especially Manoukian (1986*a*)]

$$\begin{aligned} &\langle N; N_1, N_2, \ldots, y_2^0 | M; M_1, M_2, \ldots, y_1^0 \rangle^\eta \\ &= (iU_{\alpha_1})^{N_1} (iU_{\alpha_2})^{N_2} \cdots \langle 0_+ | 0_- \rangle^\eta \cdots (iU^*_{\alpha_2})^{M_2} (iU^*_{\alpha_1})^{M_1} \end{aligned} \tag{35}$$

for given sources η, $\bar{\eta}$. Note the ordering of the U's in (35), and also note that $N_i, M_i = 0$ or 1.

For arbitrary spins $s > 0$, we introduce a symmetric source (Schwinger, 1970; see also Bargmann and Wigner, 1948) $\eta_{\alpha_1\alpha_2\ldots\alpha_{2s}}(x)$ having $2s$ spinor indices, where $\{\eta(x), \eta(x')\} = 0$ for s half-integer and $[\eta(x), \eta(x')] = 0$ for s integer. The vacuum-to-vacuum transition amplitude may be written in the compact form (Schwinger, 1970)

$$\langle 0_+ | 0_- \rangle^\eta = \exp i \int (dx)\, (dx')\, \bar{\eta}(x) S_+(x - x') \eta(x) \tag{36}$$

$$S_+(x-x') = i\int \frac{d^3\mathbf{p}}{(2\pi)^3 2p^0} e^{ip(x-x')} \prod_{\alpha=1}^{2s} [-\gamma p + m]_\alpha \qquad \text{for } x^0 > x'^0 \quad (37)$$

By considering the causal arrangement $\eta = \eta_1 + \eta_2 + \eta_3$ and repeating the analysis in Section 2, we arrive at the transition amplitudes for arbitrary spins: For s integer, the latter are given in (21) and (14), where the a_y are *replaced* by $U(y, \lambda, r)$ and the y_i, y_i' are replaced, respectively, by $\alpha = (y_2^0, \mathbf{y}, \lambda, r)$, $\alpha' = (y_1^0, \mathbf{y}, \lambda, r)$, $\lambda = -s, -s+1, \ldots, -1, 0, 1, \ldots, s$. For s half-integer, the latter (connected) amplitude is given in (35), with U_α as defined below. The $U(y, \lambda, r)$ are given by

$$U(y, \lambda, +) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2}\left[\frac{(2m)^{2s}}{2p^0}\right]^{1/2} \bar{u}_\lambda(p)\eta(p)\, e^{ipy} \quad (38)$$

$$U(y, \lambda, -) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\left[\frac{(2m)^{2s}}{2p^0}\right]^{1/2} \bar{\eta}(-p)v_\lambda(p)\, e^{ipy} \quad (39)$$

and similar expressions for $U(y, \lambda, +)^*$, $U(y, \lambda, -)^*$. For s half-integer, $(U(y, \lambda, r))^2 = 0$.

4. EXAMPLES

For the connected process $M_1, M_2, \ldots \to N_1, N_2, \ldots$, where $N_1 + N_2 + \cdots = N$, $M_1 + M_2 + \cdots = M$, in the presence of an intervening source $K(x)$, the corresponding transition amplitude is readily read from (21), (14) to be

$$\langle N; N_1, N_2, \ldots, y_2^0 | M; M_1, M_2, \ldots, y_1^0\rangle_c^K$$
$$= \frac{(ia_{y_1})^{N_1}}{(N_1!)^{1/2}} \frac{(ia_{y_2})^{N_2}}{(N_2!)^{1/2}} \cdots \langle 0_+ | 0_-\rangle^K \frac{(ia_{y_1}^*)^{M_1}}{(M_1!)^{1/2}} \frac{(ia_{y_2}^*)^{M_2}}{(M_2!)^{1/2}} \cdots \quad (40)$$

where a_y is defined in (3), (5), since in this case $m_{11}, m_{12}, \ldots, m_{21}, m_{22}, \ldots = 0$, and hence $m_1, m_2, \ldots = 0$. Of particular interest is the decay process particle $\to$ arbitrary number of particles. The latter amplitude, with the particles at specified positions, is

$$\frac{(ia_{y_1})^{N_1}}{(N_1!)^{1/2}} \frac{(ia_{y_2})^{N_2}}{(N_2!)^{1/2}} \cdots \langle 0_+ | 0_-\rangle^K (ia_{y_1}^*) \quad (41)$$

and without loss of any generality the decaying particle is chosen to be initially at time-space coordinate $(y_1^0, \mathbf{y}_1)$. The corresponding transition *probability* for the decay process where the particles (final products) emerge into a cone C: $\mathbf{x} = (r, \theta, \phi)$, $0 \le r < \infty$, $\theta_0 \le \theta \le \theta_0 + \Delta\theta$, $\phi_0 \le \phi \le \phi_0 + \Delta\phi$, where θ_0, ϕ_0 are some fixed values, is then, from (41),

$$\sum_{N=0}^{\infty} \sum_{(n_1+n_2+\cdots=N)} \frac{(|a_{x_1}|^2)^{n_1}}{n_1!} \frac{(|a_{x_2}|^2)^{n_2}}{n_2!} \cdots |\langle 0_+ | 0_-\rangle^K|^2 |a_{y_1}|^2 \quad (42)$$

where $x_i = (y_2^0, \mathbf{x}_i)$, *and* $\mathbf{x}_1, \mathbf{x}_2, \ldots$ are space points lying within the cone C. Hence we may rewrite (42) as

$$|a_{y_1}|^2 \sum_{N=0}^{\infty} \frac{(\sum_{\mathbf{x}\in C} |a_{\mathbf{x}}|^2)^N}{N!} |\langle 0_+ | 0_- \rangle^K|^2$$
$$\equiv d^3\mathbf{y}_1 \, |a(y_1)|^2 \exp\left[\int_C d^3\mathbf{x} \, |a(x)|^2\right] \exp\left[-\int_{R^3} d^3\mathbf{y} |a(y)|^2\right] \tag{43}$$

for the transition probability associated with the decay process in question. Note that the integral

$$\int_C d^3\mathbf{x} \, |a(x)|^2$$

is time (y_2^0) dependent. We are interested in the limit $y_2^0 \to \infty$, when the emerging particles, within the cone, are far from the "interaction" region. We may then invoke the classic Araki–Haag–Ruelle theorem (Araki, 1962; Ruelle, 1962; Dollard and Velo, 1966; see also Dollard, 1969; Manoukian, 1988) to write for the decay process

$$\lim_{y_2^0 \to \infty} d^3\mathbf{y}_1 \, |a(y_1)|^2 \exp\left[-\int_{R^3} d^3\mathbf{y} \, |a(y)|^2\right] \exp\left[\int_C d^3\mathbf{x} \, |a(x)|^2\right]$$
$$= d^3\mathbf{y}_1 \, |a(y_1)|^2 \exp\left[-\int_{R^3} d^3\mathbf{y} \, |a(y)|^2\right] \exp\left[\int_C \frac{d^3\mathbf{k}}{(2\pi)^3 2k^0} |K(k)|^2\right] \tag{44}$$

where

$$K(k) = \int (dx) \, K(x) e^{-ikx}, \qquad k^0 = (\mathbf{k}^2 + m^2)^{1/2} \tag{45}$$

justifying rigorously the physically expected result that the particles emerging within the cone have their momenta directed within the same cone! Note that for the $\mathbf{k}$ integration in (44), C: $\mathbf{k} = (|\mathbf{k}|, \theta, \phi)$, $0 \le |\mathbf{k}| < \infty$, $\theta_0 \le \theta \le \theta_0 + \Delta\theta$, $\phi_0 \le \phi \le \phi_0 + \Delta\phi$. Other processes are similarly studied.

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